Int J Theor Phys (2008) 47: 850–857
DOI 10.1007/s10773-007-9509-9

Torus Knots and Links from Eikonal Equations and Knot Invariants for Classification of Atoms

E.A. Elrifai

Received: 18 April 2007 / Accepted: 25 July 2007 / Published online: 3 October 2007

Abstract The history of knot theory and physics has a deep roots. It started by Lord Kelvin, in 1867, when he conjectured that atoms were knotted vortex tubes of ether. In 1997, Faddeev and Niemi suggested that knots might exist as stable soliton solution in a simple three dimensional classical field theory. That opening up a wide range of possible applications in physics. In this work we consider the Eikonal equation, which is a partial differential equation describing the traveltime propagation, which is an important part of seismic imaging algorithms. We will follow the work of Wereszczynski of solving the Eikonal equation in cylindrical coordinates. We show that only torus knots and links do occur, so figure eight knot does not occur. We show that these solutions are not unique, which means the possible occurrence of the same knot type for different configurations. Using the idea of framed knots, it is shown that two Eikonal knots are equivalent if and only if they are ambient isotopic as a framed knots, i.e. if and only if they are of the same knot type and of the same twisting number.

Keywords Solitons · Knots and Links · Braids

1 Introduction

Soliton type solutions of nonlinear field theories nonlinear field theories are of importance in various fields of physics [1]. In some cases, the space time $R \times R^n$ is topologically equivalent to $R \times S^n$, for a dimension n of space, while solitons are maps on S^n into some higher dimensional spaces, target spaces. Hence solitons may be characterized by a topological

E.A. Elrifai (✉)
Department of Mathematics, Faculty of Science, Mansoura University, P.O. Box 35516, Mansoura, Egypt
e-mail: rifai56@hotmail.com

Present address:
E.A. Elrifai
Department of Mathematics, Faculty of Science, King Khaled University, P.O. Box 9004, Abha, Saudi Arabia

Fig. 1 Positive crossing (*left*) and negative crossing (*right*)

index. For example, if the target space is a sphere S^n then the maps are characterized by the elements of the homotopy group $\pi_d(S^n)$ of order d. In fact topological solitons are of great importantce in a number of areas, including particle physics, cosmology and condensed matter physics [2, 3]. Roughly speaking, a topological soliton is a solution of a system of partial differential equation (or alternatively, a quantum field theory), these solutions are homotopically distinct according to the boundary conditions. The topological solutions with a nontrivial value of the Hopf invariant, known as knotted solitons, are of a big importance in science [4, 5]. In particular the vortex-material is defined everywhere, and its particular value defines the core. For example, in the case of a tornado, hurricane, or maelstrom the velocity vector is defined everywhere and the vortex is associated to a special point in its distribution (e.g., the eye of hurricane).

In fact, theory of knots has a long history in mathematics and physics. In 1869, knots were proposed as models of atoms, Lord Kelvin [6] is the first who proposed that atoms, then considered to be elementary particles, could be described as knotted vortex tubes in ether, so classification of knots would then give classification of atoms. Precisely, he suggested that (thin) vortex filaments can be described as a loop wrapping around the surface of a doughnut (torus) a number of times should be stable. That conjecture on the stability of torus knots still remains unproven. Now the possibility of knotted elementary structures has been considered again. But in order to give knots a physical meaning one has to give them some physical properties, such as energy and thickness. For example, one may assume that there is a charge distribution on the wire and then find its minimum energy configuration.

One of the bridges from physics to knots, is to find a theoretical model where knots emerge as solitons, i.e. as stable finite energy solutions to the pertinent nonlinear field equations. In 1975, Faddeev [7, 8] proposed that closed knotted vortices could be constructed in a finite dynamical model, his model is quite universal. In [9] Faddeev and Niemi, constructed knotlike vortices for the model in [7, 8], they found indications supporting the existence of the unknot and trefoil vortices. Also they conjectured that all torus knots should appear as vortex solitons.

The classical field theory with a unit vector field defined everywhere is a one of the main bridges between physics and knot theory. For example, the unit vector can be considered as a point in S^2, and the vector field provides a map $\vec{n}: R^3 \approx S^3 \to S^2$. Such maps can be labeled by their Hopf charge. But, in general, the equations of motion that describe knotted solitons are highly complex nonlinear partial differential equation.

2 Knots, Links and Braids

A knot is an embedded circle in $S^3(R^3)$, and a link is a disjoint collection of knots. The unknot is a trivial circle $S^1 \subset S^2 \subset S^3$. Two links $L_1$and L_2 are ambient isotopy if there is a homotopy $H_t : S^3 \longrightarrow S^3, t \in [0, 1] = I$, such that H_0 is the identity map, H_1 sends L_1 to L_2 and H_t is a homeomorphism for every $t \in I$. So knots and links are in the same knot and link type if and only if they are equivalent under ambient isotopy.

All knots and links in this paper are oriented by the arrow of time. Diagramatically, knots and links are described geometrically via projections onto a plane in which over and under

Fig. 2 The geometric braid σ_i

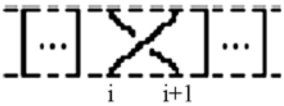

Fig. 3 A braid word σ_1^3 (*left*). The trefoil as closed braid (*right*)

crossings are indicated by broken lines. In each crossing in a knot diagram, one of the two pictures in Fig. 1 will hold, count a $+1$ for the first and a -1 for the second. One can recognize it by considering the oriented over arc as the positive direction of the x-axis and the under arc as the y-axis, then the crossing counts $+1$ or -1 if y-axis in positive or negative direction, respectively.

A knot K is a composite knot if it is obtained by connecting diagrams of two nontrivial knots K_1 and K_2, where a new knot is obtained by removing a small arc from each knot projection and then connecting the four endpoints by new arcs, the prime knot is a noncomposite knot.

Linking number of a link L of two components M and N is the division by 2 of the sum of the $+1$s and -1s, as in Fig. 1, over all the crossings between M and N, denoted $lk(M,N)=lk(L)$. The link L of n components $K_1, K_2, \dots, K_n$, has linking number $lk(L)=\sum_{1\le i\prec j\le n} lk(K_i,K_j)$.

A (p,q)-torus knot is obtained by looping a string through the hole of a torus p-times with q-revolutions before joining its ends, where p and q are relatively prime. A (p,q)-torus knot is equivalent to a (q,p)-torus knot. All torus knots are prime, chiral and invertible.

The general n-braid group, B_n, has the presentation

$$\{\sigma_i, i=1,2,\dots,n-1 \mid \sigma_i\sigma_j=\sigma_j\sigma_i, |i-j|\ge 2, \sigma_i\sigma_{i+1}\sigma_i=\sigma_{i+1}\sigma_i\sigma_{i+1}, i\le n-2\},$$

where the generators can be represented geometrically as in Fig. 2, the closure (closed braid) of a braid $\alpha\in B_n$ formed by joining the top points to the bottom and denoted $\widehat{\alpha}$, for $\alpha=\sigma_1\sigma_2\sigma_1\sigma_2\in B_n$, $\widehat{\alpha}$ is the trefoil knot, as shown in Fig. 3. Two closed braid are equivalent (as links) if and only if the braids are related by a finite sequence of the following Markov moves:

$$M1(\text{conjugation}): b \longleftrightarrow aba^{-1} \text{ for any } a,b\in Bn,$$

$$M2(\text{stabilizar}): b \longleftrightarrow b\sigma_n^{\pm 1} \text{ for any } b\in Bn.$$

For more details about braids theory and link theory, we refer to [10] and [11].

3 Eikonal Equations

The Eikonal differential equation is the basic mathematical model, describing the traveltime (also called the Eikonal) propagation in a given velocity model. Traveltime computation is an important part of seismic imaging algorithms. The general form of the nonlinear Eikonal equation is

$$(\nabla u)^2 = \sum_{i=1}^{n} \left(\frac{\partial u}{\partial x_i} \right)^2 = F^2 t(x), \tag{1}$$

where $u(x)$ is the traveltime from the source $\Omega \subset R^2$ to the point x, and $F(x)$ is the slowness at that point. Now consider the complex isotropic ($F = 0$) Eikonal equation,

$$(\nabla u)^2 = 0 \tag{2}$$

in $(2 + 1)$ or $(3 + 1)$ dimensional space-time, where u is a complex scalar field. Such a field can be related, by means of the standard, stereographic projection with a unit three-components vector field $\vec{n} \in S^2$,

$$\vec{n} = \frac{1}{1 + |u|^2}(u + u^*, -i(u - u^*), |u^2| - 1). \tag{3}$$

This vector field defines the topological contents of the model. Depending on the number of space dimensions and asymptotic conditions this field can be treated as a map with $\pi_2(S^2)$ or $\pi_3(S^2)$ topological charge. The most important case for us is the $(3 + 1)$ dimensional case, where in [12] Wereszczynski has found a multi-knot configurations with an arbitrary value of the Hopf index. In such these cases, it is not sufficient to know the location of the vortex core, which is the preimage of the south pole of the map $\vec{n}: R^3 \approx S^3 \to S^2$, but also the twisting around the core. So that the proper knot theoretical setting for the analysis of knots associated to unit vector fields is to use framed links.

4 Torus Knots and Links from Eikonal Equations

The Eikonal equation (2) in cylindrical coordinates takes the form

$$(\partial_\rho u)^2 + \frac{1}{\rho^2}(\partial_\phi u)^2 + (\partial_z u)^2 = 0. \tag{4}$$

Then using the Ansatz

$$u(\rho, \phi, z) = R(\rho)\Phi(\phi)Z(z), \tag{5}$$

we have

$$\frac{1}{R^2}(\partial_\rho R)^2 + \frac{1}{\rho^2 \Phi^2}(\partial_\phi \Phi)^2 + \frac{1}{Z^2}(\partial_z Z)^2 = 0, \tag{6}$$

which can be rewritten as a system of ordinary differential equations

$$Z'^2 + k^2 Z^2 = 0, \qquad \Phi'^2 + n^2 \Phi^2 = 0, \qquad R'^2 - \left(\frac{n^2}{\rho^2} + k^2 \right) R^2 = 0. \tag{7}$$

Thus

$$Z(z) = Ae^{\pm ikz}, \qquad \Phi(\phi) = Be^{\pm in\phi}, \qquad R(\rho) = D\left(\frac{\rho}{n + \sqrt{k^2\rho^2 + n^2}} \right)^{\pm n} e^{\pm\sqrt{k^2\rho^2+n^2}}, \tag{8}$$

where A, B and D are complex constants, k^2 is a positive constant and for uniqueness of solution n must be an integer. Hence, this can derive the following solution of the Eikonal equation,

$$u(\rho,\phi,z) = C\left(\frac{\rho}{n+\sqrt{k^2\rho^2+n^2}}\right)^{\pm n} e^{\pm\sqrt{k^2\rho^2+n^2}} e^{\pm i(n\phi+kz)}. \tag{9}$$

Also we can yield a more general solution, as a sum of the above solution

$$u(\rho,\phi,z) = \sum_{j=1}^{N} C_j\left(\frac{\rho}{n_j+\sqrt{k_j^2\rho_j^2+n_j^2}}\right)^{\pm n_j} e^{\pm\sqrt{k_j^2\rho_j^2+n_j^2}} e^{\pm i(n_j\phi+k_jz)} + c_0, \tag{10}$$

where c_0 is a constant and also n_j is an integer and $k_j/n_j = const.$ for every j. Notice that the cylindrical solution in (10), and its generalization in (11), implies that $\lim_{\rho\to\infty} \vec{n} = (0,0,1)$. So from (2), this equivalent to $|u^2| = 0$. Then if we considered the simple case, where $C_i = 1$ and c_0 is real, the condition $|u^2| = 0$, will imply the following algebraic equation,

$$\sum_{i,j=1}^{N} R_i R_j \cos[(k_i-k_j)z + (n_i-n_j)\phi] + 2c_0\sum_{i=1}^{N} R_i\cos[k_iz+n_i\phi] + c_0^2 = 0. \tag{11}$$

Now let us review the solution of the above equation in cases when $N = 1, 2, \ldots,$ [12].

Theorem 1 *All torus knots and links do occur.*

Proof We are going to analyse (11) in different values of N, with arbitrary values of n and k. In case when $N = 1$, this equation takes the form

$$R^2 + 2c_0R\cos[kz+n\phi] + c_0^2 = 0, \tag{12}$$

then solving this quadratic equation of R, we have

$$R(\rho) = -c_0 e^{\pm i(kz+n\phi)}, \tag{13}$$

where $R(\rho_0) = c_{0,}$ gives a cylinder with constant radius $\rho = \rho_0$, which implies that $e^{\pm i(kz+n\phi)} = -1$, and the solutions are n strings (arcs) on that cylinder, that for $kz + n\phi = \pi + 2l\pi$, with $l = 0, 1, 2, \ldots, n-1$. This gives

$$z = \frac{(2l+1)\pi - n\phi}{k}, \quad l = 0, 1, 2, \ldots, n-1. \tag{14}$$

Now, let us start at the level $z = 0$, then from (14), we have $\phi = \frac{\pi}{n}, \frac{3\pi}{n}, \ldots, \frac{(2n-1)\pi}{n}$. But we have k coils on each interval with length $2n\pi$ on z-axis. This means that if we rotate a circle at level $z = 0$ with fixed n points $(\rho_0, \frac{2i-1}{n}\pi, 0)$, $i = 1, 2, \ldots, n$ on it, we will have a parallel circle with the same radius ρ_0 and at $z = \frac{2\pi}{nk}$, with n points $(\rho_0, \frac{2i-1}{n}\pi, \frac{2\pi}{nk})$, $i = 2, 3, \ldots, n, 1$, after an action on the circle by a rotation with angle $\phi = \frac{\pi}{n}$. Hence each rotation with angle $\phi = \frac{2\pi}{nk}$ means a one coil. Therefore, for an arbitrary n and for a fixed plane $z = z_0$, the n points in that plane are $(\rho_0, \frac{1}{n}((2\pi l + 1)\pi - z_0), z_0)$, $l = 0, 1, \ldots, n-1$. Similarly, for $N = 2$, we have one central string at $\rho = 0$, which representing an axis of the other strings. So that we are in case of a link. □

Corollary 2 *The corresponding closed braid representation of the obtained Eikonal link* (*knot*) *is* $(\sigma_1\sigma_2\cdots\sigma_{n-1})^k \in B_n$.

Proof It is a direct consequence of Theorem 1, since the jumping of the parameter z from level $z = z_0$ to the next level $z = z_0 + \frac{2\pi}{k}$, this means a permutation $(\rho_0, \frac{1}{n}((2\pi l + 1)\pi - z_0), z_0) \rightarrow (\rho_0, \frac{1}{n}((2\pi l + 1)\pi - z_0) + \frac{\pi}{n}, z_0 + \frac{2\pi}{k})$. Also the strings are monotonically increasing and in a symmetric rotation with respect to the changes of z, on a cylinder with constant radius. So, if we take a portion of the cylinder with k-coils, then each one coil has the permutation $i_1 i_2 i_3 \cdots i_{n-1} i_n \rightarrow i_2 i_3 \cdots i_{n-1} i_n i_1$, which represent the braid $\sigma_1\sigma_2\cdots\sigma_{n-1}$. So the k-coils give the braid $(\sigma_1\sigma_2\cdots\sigma_{n-1})^{nk}$. But the torus link of type (m, n) is the closure of the braid $(\sigma_1\sigma_2\cdots\sigma_{n-1})^m \in B_n, m, n \in N$, and the link is a knot if and only if m and n are relatively prime, while the torus link of type (n, n) has the closed braid representative $\Delta_n^2 = (\sigma_1\sigma_2\cdots\sigma_{n-1})^n \in B_n$, which is the generator of the center of the group B_n. □

Lemma 3 *Eikonal knots and links are not unique*, *it may occurs many times*.

Proof We will show that we may have a knot or a link with many different closed braid representative, i.e. with different values of k and n, let us take $N = 1$:

(a) *The unknot*: There are an infinite number of the closed braid representatives of the unknot, where we can take $k = \frac{1}{n}$ for every $n \in N$, then $((\sigma_1\sigma_2\cdots\sigma_{n-1})^n)^{\frac{1}{n}} = (\sigma_1\sigma_2\cdots\sigma_{n-1}) \in B_n$ is a closed braid representative for the trivial knot.

(b) *The trefoil knot*: (i) For $n = 2, k = \frac{3}{2}$, the corresponding configuration has the closed braid representative $(\Delta_2^2)^{\frac{3}{2}} = ((\sigma_1^2))^{\frac{3}{2}} = \sigma_1^3 \in B_2$.

(ii) For $n = 3$, $k = \frac{2}{3}$, the corresponding configuration has the closed braid representative $(\Delta_3^2)^{\frac{2}{3}} = ((\sigma_1\sigma_2)^3)^{\frac{2}{3}} = (\sigma_1\sigma_2)^2 \in B_3$, by using braid operations, we find that $(\sigma_1\sigma_2)^2 = \sigma_1\sigma_2\sigma_1\sigma_2 = \sigma_1\sigma_1\sigma_2\sigma_1 = \sigma_1^2\sigma_2\sigma_1$, which is conjugate to $\sigma_1^3\sigma_2$, hence removing the trivial curl by applying the stabilizer Markov move, it will be $\sigma_1^3 \in B_2$.

(c) *The knot* 5_1: (i) For $n = 2$, $k = \frac{5}{2}$, the corresponding configuration has the closed braid representative $(\Delta_2^2)^{\frac{5}{2}} = ((\sigma_1^2))^{\frac{5}{2}} = \sigma_1^5 \in B_2$.

(ii) For $n = 5$, $k = \frac{2}{5}$, the corresponding configuration has the closed braid representative $(\Delta_5^2)^{\frac{3}{5}} = ((\sigma_1\sigma_2\sigma_3\sigma_4)^5)^{\frac{2}{5}} = (\sigma_1\sigma_2\sigma_3\sigma_4)^2 \in B_5$, where

$$
\begin{aligned}
(\sigma_1\sigma_2\sigma_3\sigma_4)^2 &= \sigma_1\sigma_2\boxed{\sigma_3\sigma_4\sigma_1\sigma_2}\sigma_3\sigma_4 = \sigma_1\sigma_2\boxed{\sigma_1\sigma_3\sigma_2\sigma_4}\sigma_3\sigma_4 \\
&= \boxed{\sigma_1\sigma_2\sigma_1}\underline{\sigma_3\sigma_2\sigma_4\sigma_3}\sigma_4 = \boxed{\sigma_2\sigma_1\sigma_2}\underline{\sigma_3\sigma_2\sigma_3\sigma_4}\sigma_3 \\
&= \sigma_2\sigma_1\sigma_2\sigma_3\sigma_2\sigma_3\sigma_4\boxed{\sigma_3} \overset{\text{conjugation}}{=} \boxed{\sigma_3}\sigma_2\sigma_1\sigma_2\sigma_3\sigma_2\sigma_3\underline{\sigma_4} \overset{\text{stabilizar}}{=} \sigma_3\sigma_2\sigma_1\sigma_2\sigma_3\sigma_2\sigma_3 \\
&= \sigma_3\sigma_2\sigma_1\boxed{\sigma_2\sigma_3\sigma_2\sigma_3} \overset{\text{conjugation}}{=} \boxed{\sigma_2\sigma_3\sigma_2\sigma_3}\sigma_3\sigma_2\underline{\sigma_1} \overset{\text{stabilizar}}{=} \boxed{\sigma_2\sigma_3\sigma_2}\sigma_3\sigma_3\sigma \\
&= \boxed{\sigma_3\sigma_2\sigma_3}\sigma_3\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3^3\boxed{\sigma_2} = \underline{\boxed{\sigma_2}\sigma_3\sigma_2}\sigma_3^3 = \underline{\sigma_3\sigma_2\sigma_3}\sigma_3^3 \\
&= \sigma_3\sigma_2\sigma_3^4 \overset{\text{conjugation}}{=} \sigma_3^5\sigma_2 \overset{\text{stabilizar}}{=} \sigma_3^5.
\end{aligned}
$$

□

5 Eikonal Knots and Links as a Framed Knots and Framed Link

If we take a strip of paper and twist it, possibly tie a knot on it, and then glue its ends together. Then we obtain a closed twisted and possibly knotted strip, or a framed knot. On a framed knot we are interested in its boundary curve(s) and in its center curve. Closed strips are of two kinds, either of even or odd number of half twists, where a half twist as in Fig. 4. In each strip we have a center curve, $c_{1,}$ and a strip with even number of half twists has two boundary curves, c_2, c_3, while in case of a strip with an odd number of half twists, we half a one boundary curve, c_2.

Definition 4 Let M be a closed strip in R^3, then the twisting number of M, $Tw(M)$, is given by the formula $Tw(M) = \frac{1}{2} lk(c_1, c_2)$, for the strip of odd number of half twists, and $Tw(M) = \frac{1}{2}\{lk(c_1, c_2) + lk(c_1, c_3)\}$, for the strip of even number of half twists.

From the above definition, there is a one to one correspondence between integers, Z, and class of unknotted closed strips with even number of half twists. Also here is a one to one correspondence between the set $\{\frac{2n+1}{2}, n \in Z\}$ and class of unknotted closed strips with even number of half twists [13].

Theorem 5 [13] *Closed strips are classified by their twisting number and by the link type given by the centre curves of the closed strips.*

Proposition 6 *Eikonal knots are distinguished by their knot type and by the twisting number of the associated framed knot.*

Proof The proof is a direct consequence of Theorem 5. It is known that for coprime integers p, q, the torus knots (p, q) and (q, p) are of the same knot type. But as a framed knots they are different, since they have twisting numbers $p(q-1)$ and $q(p-1)$ respectively, where each crossing contributes 1 for the twisting number. So $p(q-1) \neq q(p-1)$ otherwise $p = q$. □

Remark 1 As in Lemma 3, the Eikonal unknotted configurations have closed braid representatives $(\sigma_1\sigma_2 \cdots \sigma_{n-1}) \in B_n$, and the associated framed knot has twisting number $\frac{1}{2}(n-1)$ for every integer $n \geq 2$. Also the trefoil with braid representative $\sigma_1^3 \in B_2$ has twisting number 3, but the trefoil with braid representative $(\sigma_1\sigma_2)^2 \in B_3$ has twisting number 3. But the 5_1 knot with braid representative $\sigma_1^5 \in B_2$ has twisting number 5, while the 5_1 knot with braid representative $(\sigma_1\sigma_2\sigma_3\sigma_4)^2 \in B_5$ has twisting number 8.

6 Speculations

According to Lemma 3, and applying Theorem 5, in fact torus knots are a widely studied class of space curves, convenient because of the surface on which they lie and the natural way in which they are defined. So I think that it will be interesting to apply some geometric invariants of space curves, specially these invariants which combine between the embedding of the curve on some surface and the embedding of the surface itself. For instance, it is well known that two closed embedded curves in R^3 are equivalent as knots if and only if they are ambient isotopic, the two curves in Fig. 5, are equivalent to the trivial knot. But if we constructed these two curves so that they are of class C^3 and have nonvanishing curvature at each point, then these two curves have different self-linking number. Properly the self-linking number for a (p, q) torus knot changes as the rigid geometric structure of the torus on which the curve lies is changed, the $(3, 2)$ knot has self-linking number 3 and the $(2, 3)$ knot has self-linking number 4. Where a curve in three-space is said to have self-linking

Fig. 4 A strip with half twist

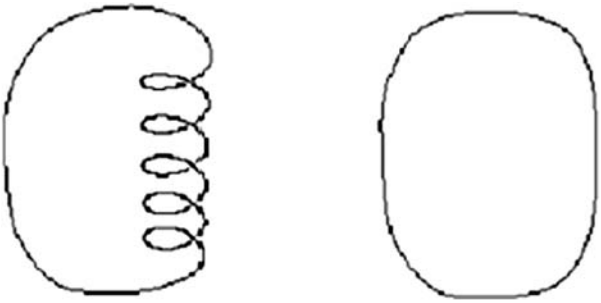

Fig. 5 Two equivalent unknots

number n if the curve has linking number n with the push-off curve in the direction of the principal normal vector [14].

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